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Geometrical and topological aspects of Electrostatics on Riemannian manifolds

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Abstract

We study some geometrical and topological properties of the electric fields created by point charges on Riemannian manifolds from the viewpoint of the theory of dynamical systems. We provide a thorough description of the structure of the basin boundary and its connection with the topology of the manifold, and characterize the spaces in which the orbits of the electric field are geodesics. We also consider symmetries of electric fields on manifolds, particularly on spaces of constant curvature.

Keywords: Green function; Vector field; Basin boundary; Geodesics; Symmetries

1. Introduction

The discovery of the inverse-square law for Newtonian and Coulomb interactions is a milestone in the physics of the seventeenth and eighteenth centuries. The central claim of electrostatic theory [2,24] is that the force per unit charge experienced by a test particle situated at a point $x \in \mathbb{R}^3$ subject to the interaction created by a charge of magnitude $q \in \mathbb{R}$ is given by the electric vector field

$$E = \frac{q}{4\pi} \frac{x - x_0}{|x - x_0|^3}.$$

Here $x_0 \in \mathbb{R}^3$ is the position of the point particle originating the interaction, and we have chosen Heaviside–Lorentz units. The same law also holds for the gravitational interaction created by a point mass of magnitude $-q$ in natural units.

Since then, the study of electric fields generated by N point charges q_i $(i = 1, \dots, N)$ in Euclidean space has become a classical problem in mathematical physics and potential theory [11]. When the charges are all negative, this is equivalent to studying the Newtonian gravitational field created by N point masses $|q_i|$, which also coincides with the first-order approximation to the gravitational field in general relativity [37]. In modern treatments, one usually

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defines the potential function $V_p : \mathbb{R}^3 \to \mathbb{R}$ of a point charge, which is a fundamental solution of the Poisson equation

$$-\Delta V_p = \delta_p,$$

and obtains the electric field as $E = -\nabla V_p$. Here and in what follows, δ_p stands for the Dirac distribution centered at p. The electric field created by several charges can be calculated using the superposition principle.

A natural generalization of this problem is the study of the electric fields generated by point charges on Riemannian spaces. There is a vast literature on the fundamental solutions of the Poisson equation on manifolds, e.g., on the existence of positive fundamental solutions [33,9,17,29,30], the study of upper and lower estimates for these functions [45,31,22], and the connection of these fundamental solutions with the heat kernel [51,32,18].

Nevertheless, the geometric and topological properties of the gradient of the fundamental solutions have received comparatively little attention. In this paper we shall focus on the study of this aspect using techniques from the theory of dynamical systems, and we shall show some interesting connections between the orbits of the electric field (historically known as electric lines or lines of force) and the topology of the space. Thus the concept of electric line, as Faraday used to visualize the electric fields in the nineteenth century, is profitably extended to the framework of general Riemannian manifolds.

Let us sketch the organization of this paper. In Section 2 we define the concepts of Li–Tam fundamental solution, basin boundary, and some other objects of which we make extensive use in the following sections. In Section 3 the topological structure of the electric lines and the basin boundary in an n-manifold is studied, whereas in Section 4 we provide stronger results which hold for electric fields on surfaces ($n = 2$). Section 5 concentrates on the relationship between electric lines and geodesics. In Section 6 we study the symmetries of the electric field and their application to spaces of constant curvature, obtaining some exact solutions. Most of the material in Sections 3–6 is new, including a detailed description of the topological structure of the basin boundary, and a complete characterization of spaces in which the electric lines are geodesics.

2. Definitions

Let (M, g) be a Riemannian n-manifold without boundary, which we shall assume to be open, complete, analytic, connected, finitely generated (i.e., all the homotopy groups of M have finite rank), and such that all its ends are collared. For an arbitrary point $p \in M$, let V_p be a fundamental solution of the Poisson equation

$$-\Delta V_p = \delta_p, \tag{1}$$

Δ standing for the Laplace–Beltrami operator. Here δ_p denotes the Dirac distribution centered at p.

Li and Tam [29] have provided a geometric construction of solutions to this equation for any Riemannian manifold (M, g). Their technique consists in considering a monotone sequence of compact domains $p \in M_1 \subset M_2 \subset \cdots$ which exhaust M, and studying the Dirichlet problem

$$-\Delta V_p^{(k)} = \delta_p \quad \text{in } M_k$$
$$V_p^{(k)} = 0 \quad \text{on } \partial M_k$$

in each M_k. Then a solution to Eq. (1) can be obtained as

$$V_p(x) = \lim_{k\to\infty} V_p^{(k)}(x) - c_k$$

for some sequence of non-negative constants (c_k). The construction guarantees that V_p is analytic in $M - p$, and that it is decreasing in the sense that for all $R > 0$

$$\sup_{M-B_p(R)} V_p = \max_{\partial B_p(R)} V_p,$$

where $B_p(R) = \{x \in M : \operatorname{dist}(x, p) < R\}$. These two properties are key to most of our work in the following sections. Furthermore, the map $v : M \times M \to \mathbb{R}$ given by $v(x, y) = V_y(x)$ is symmetric, and analytic in $\{(x, y) \in M \times M : x \neq y\}$.

When $\inf V_p = -\infty$, V_p is called a non-positive Green function, or an Evans function. This condition only depends on the end structure of (M, g), and when it holds (M, g) is called parabolic. When $\inf V_p > -\infty$, one says that

V_p is a (positive) Green function, and (M, g) is called hyperbolic. There is an extensive literature on geometric conditions characterizing hyperbolic and parabolic spaces, e.g., [9,17,31,22]. When the manifold is hyperbolic, Li and Tam's construction provides the unique minimal positive fundamental solution. Otherwise, uniqueness is not usually guaranteed, except for some particular cases [29].

A configuration of point charges on M is a set $\mathcal{C} = \{(q_i, p_i)\}_{i=1}^N$, where N is the number of charges, and $(q_i, p_i) \in (\mathbb{R} - 0) \times M$ represents the magnitude and position of the i-th charge of the configuration.

Definition 1. The electric field E created by the charge configuration $\mathcal{C}$ is defined as $E = -\nabla V$, where the potential function V is given by

$$V = \sum_{i=1}^{N} q_i V_{p_i},$$

$V_p = v(\cdot, p)$, and $v : M \times M \to \mathbb{R}$ stands for a fixed solution to Eq. (1) obtained via Li and Tam's procedure.

Obviously the electric field is an analytic, divergence-free vector field on $M - \bigcup_{i=1}^N p_i$ satisfying Maxwell's equations on the manifold. Moreover, its critical set has codimension greater than 1 as a consequence of the Cauchy–Kowalewski theorem. The positions of the charges are clearly the only singularities of the electric field (i.e., $\lim_{x \to p_i} |E(x)| = \infty$). Observe that the definition of the electric field does not require (M, g) to be hyperbolic. In fact, since we will be interested in the properties of the orbits of E (which from now on will be called *electric lines*), the hyperbolicity or parabolicity of the manifold will not be especially relevant. Actually, recall that even the Euclidean plane $(\mathbb{R}^2, \delta)$ is a parabolic space. One should also note that Li and Tam's solutions to Eq. (1) are physically admissible in both cases, since they are symmetric and decreasing.

Let $\mathcal{C}$ be a configuration of negative point charges. Two key objects in the study of the portrait of the electric lines in the large are the *attracting basin* and the *basin boundary* of $\mathcal{C}$, which we shall now define.

Definition 2. The (attracting) basin of the charge (q_i, p_i) is

$$D_i = \{x \in M : \omega(x) = p_i\},$$

where $\omega(x)$ is the ω-limit of the orbit of E passing through x. The (attracting) basin of the configuration $\mathcal{C}$ is defined as $D = \bigcup_{i=1}^N D_i$.

Definition 3. The basin boundary of the configuration $\mathcal{C}$ is $\mathcal{F} = \partial D$.

Thus the basin D_i consists of the points that are dragged into the i-th charge along the flow of E. Being a boundary, $\mathcal{F}$ has codimension at least 1, and the electric lines passing through some point of $\mathcal{F}$ do not fall into any charge. In Section 3 we will provide a detailed characterization of these sets.

For reasons that will become apparent in Sections 3 and 4, it is technically convenient to introduce a compactification of the manifold M and of the basin boundary $\mathcal{F}$ as follows. Since M is finitely generated and all its m ends are collared, there exist [23] a closed topological n-manifold K and a finite subset $\{K_i\}_{i=1}^m$ of pairwise disjoint compact submanifolds of K such that M is homeomorphic to $K - \bigcup K_i$. We define the collared-end compactification of M as $\hat{M} = K - \bigcup \operatorname{int}(K_i) = M \cup \mathcal{E}(M)$, and its compactified boundary as $\hat{\mathcal{F}} = \mathcal{F} \cup \mathcal{E}(M)$. Here and in what follows we use the same notation for a subset of M and its homeomorphic image in $\hat{M}$, and we denote the set of ends of M by $\mathcal{E}(M) = \{\mathcal{E}_i\}$, where $\mathcal{E}_i = K_i - \operatorname{int}(K_i)$. One should also note that any topological submanifold of $\hat{M}$ not containing any end naturally inherits a Riemannian structure, but neither $\mathcal{F}$ nor $\hat{\mathcal{F}}$ is generally a topological submanifold.

Although we shall be primarily interested in point charges, we will also consider extended charge distributions, which are given by piecewise smooth functions $\rho : M \to \mathbb{R}$, possibly with compact support.

Definition 4. The electric field E created by a charge distribution ρ is $E = -\nabla V$, where the potential function $V : M \to \mathbb{R}$ is defined as

$$V(x) = \int_M v(x, y)\rho(y)\mathrm{d}y$$

when the integral exists, v being a fixed Li–Tam solution to Eq. (1).

One should observe that $-\Delta V = \rho$.

3. Topological structure of the electric lines

We shall now study the topology of the orbits of the electric field created by a configuration $\mathcal{C}$ of point charges. The structure of the electric field is not well understood as a dynamical system, either locally (e.g., the portrait near the critical points and singularities) or in its global aspects. Since we will be mainly interested in the topological properties of the basin boundary, in this section we assume that the charges are all negative.

First one should recall that the analytical local behavior of E near a singularity p_i is well known [13], and in fact can be easily obtained by direct integration. Given a chart $(x^i) : U \subset M \to \mathbb{R}^n$ centered at p_i, one can construct the expression

$$E = \frac{q_i c_n}{\sqrt{G} r^{n-1}} \partial_r + W \tag{2}$$

in U. Hereafter $r^2 = \sum_{i=1}^n (x^i)^2$, $\sqrt{G}$ is the volume density function (i.e., G is the determinant of the metric in the coordinates (x^i)), and c_n^{-1} denotes the area of the round $(n-1)$-sphere. Besides, the vector field W is divergence-free and analytic in U. Refs. [33,29] ensure that a certain W exists such that this local solution can be globally extended. In U, one can define a desingularized electric field as

$$\tilde{E} = r^n E \sim -r\partial_r + O(r^2). \tag{3}$$

In addition to $\tilde{E}$, we shall also make use of the vector field in U

$$X = \frac{1}{1 + |\tilde{E}|^2} \tilde{E}. \tag{4}$$

Note that $\tilde{E}$ and X are analytic, and possess the same orbits as E. Furthermore, X is a complete vector field in U.

When (x^i) are normal Riemann coordinates, r is the geodesic distance to p_i, and the asymptotic behavior of the metric is

$$G \sim 1 + O(r^2) \tag{5}$$

as r tends to zero. Besides, the singularities of E are Newtonian in the sense that $|E| \sim r^{1-n} + O(r^{3-n})$.

In the following proposition we gather some fundamental properties of the electric lines. Properties (1) and (2) provide a quite detailed description of the electric lines near the charges up to (local) C^ω diffeomorphism, whereas Properties (3), (4), (5) and (6) convey information on the portrait of the electric lines in the large.

Proposition 1. *For the electric field E created by the charge configuration $\mathcal{C}$, the following statements hold:*

(1) *p_i is a local attractor, and its neighboring equipotential sets $V^{-1}(c)$ ($c \in \mathbb{R}$) are topological spheres.*
(2) *Let A be an analytic subset of M. Then, in a neighborhood of p_i, the electric trajectories, which emanate from p_i, either stay in A or intersect it in a finite number of points. Hence all the orbits have well defined tangent at p_i.*
(3) *The electric trajectories point inward at infinity (i.e., the ends of M are local maxima of the potential).*
(4) *The equipotential sets are compact analytic sets of codimension 1. In particular, they have no endpoints.*
(5) *There exist no invariant closed sets without charge and with non-empty interior.*
(6) *E does not have any periodic orbits.*

Proof. (1) Since the singularity p_i is an isolated minimum of the potential ($\lim_{x\to p_i} V(x) = -\infty$), its neighboring equipotential sets are topological spheres, proving the claim.

(2) Since the linearization of $\tilde{E}$ is proportional to $r\partial_r = \sum x_i \partial_i$, the eigenvalues of its derivative are all equal, and thus Siegel's (C, ν) condition is satisfied [1]. Hence $\tilde{E}$ is locally C^ω-conjugate to its linear part. As the claim holds for the orbits of the linearization of $\tilde{E}$, and E and $\tilde{E}$ have the same orbits in a neighborhood of the singular point, the electric lines must also intersect any given analytic set a finite number of times. This implies the existence of a well defined tangent; cf. Ref. [28].

(3) Li and Tam's construction ensures that V is non-decreasing, tending to a definite limit on each end (possibly $+\infty$). Thus each end of M is a local maximum, and in fact its "neighboring" equipotential sets are tubes whenever V has no critical points outside some compact set.
(4) By analyticity of V, $V^{-1}(c)$ is an analytic set, and hence closed. As a consequence of Property (3), the equipotential sets must be bounded, proving compactness. Sullivan's theorem [50] implies that $V^{-1}(c)$ has no endpoints. To prove that the codimension is 1, let us assume there exists a point $x \in V^{-1}(c)$ such that the connected set $W = V^{-1}(c) \cap U$ has codimension greater than 1, U being a sufficiently small neighborhood of x. Then the implicit function theorem shows that W belongs to the critical set of V. Since $V \neq c$ in $U - W$, then the equipotential sets of V in U are tubes around W, and hence W is a local extremum, contradicting the harmonicity of V.
(5) Let S be a closed invariant set without charge. Since the ends of the manifold are local maxima, V must attain its minimum on S regardless of whether S is compact or not. S being invariant, the flow of X must possess a local attractor at the latter minimum, contradicting the harmonicity of $V|_S$.
(6) Being a gradient field, E cannot have periodic orbits. □

Remark 1. Property (2) states that the orbits of E are non-oscillating (i.e., E satisfies the analytic finiteness conjecture [28]) near its singular points. For $n > 2$, it is not known whether the orbits of E (more generally, of the gradient of an analytic function) are also non-oscillating at its critical points.

Let us now focus on the topology of the attracting basins and their boundary, which encloses the homological properties of the manifold. First we shall prove that each basin is diffeomorphic to $\mathbb{R}^n$.

Proposition 2. *D_i is an open, invariant submanifold of M diffeomorphic to $\mathbb{R}^n$.*

Proof. First, it should be observed that D_i deform retracts to p_i and that D_i is invariant, so its homotopy groups are trivial and D_i is homeomorphic either to $\mathbb{R}^n$ or to a Whitehead-type manifold [14]. In proving that it is indeed homeomorphic to $\mathbb{R}^n$, Property (1) in Proposition 1 implies that there exists a topological n-disc $B = \{x \in D_i : V(x) < c\}$. Let $h : M \to \mathbb{R}$ be a smooth function, positive in $M - \bigcup p_i$, which vanishes as r^n at each singularity p_i, r standing for the geodesic distance to p_i. Let ϕ_t be the flow of the complete vector field $Y = h(1 + h^2|E|^2)^{-1}E$. Then $D_i = \bigcup_{j=1}^{\infty} \phi_j(B)$, and hence by Ref. [6] D_i must be homeomorphic to $\mathbb{R}^n$. By uniqueness of analytic differentiable structures on a manifold [44], the fact that D_i is an analytic submanifold now implies that it is C^ω-diffeomorphic to $\mathbb{R}^n$. □

Theorem 1. *The following statements hold:*
(1) *The boundary $\mathcal{F}$ is a closed invariant set, and $\hat{\mathcal{F}}$ is compact.*
(2) *M is the disjoint copy of D and $\mathcal{F}$.*
(3) *The boundary is non-empty whenever M is not homeomorphic to $\mathbb{R}^n$ or there is more than one charge.*
(4) *The α-limit of an electric line contained in $\mathcal{F}$ is either a critical point or an end of the manifold, and its ω-limit must be a critical point. In particular, $\mathcal{F}$ consists of the union of the critical points of V and their stable components.*

Proof. (1) D being open, $\mathcal{F} = \overline{D} - D$ must be closed, so that $\hat{\mathcal{F}}$ is compact. $\mathcal{F}$ is clearly invariant since $\overline{D}$ and D are.
(2) Let U be an open subset of the closed set $M - D$, which we can assume to be invariant without loss of generality. Then $\overline{U}$ is a closed invariant set without charge, with non-empty interior if $U \neq \emptyset$. By Property (5) of Proposition 1, $M - D$ has empty interior, and therefore $M = \overline{D}$. Hence $M - D = \overline{D} - D = \mathcal{F}$.
(3) By Property (2), $M = D \cup \mathcal{F}$, D being homeomorphic to N disjoint copies of $\mathbb{R}^n$. When $N \neq 1$, D is not connected, and cannot be homeomorphic to M. When $M \ncong \mathbb{R}^n$, $\mathcal{F}$ cannot be empty either even if $N = 1$.
(4) Let U be the union of the critical set of E and its stable components. U is clearly invariant. To prove that $\mathcal{F}$ is contained in U, let $O \subset \mathcal{F}$ be an orbit of E. Its ω-limit cannot be a charge, since it lies on $\mathcal{F}$, and O cannot escape to infinity, since the field points inward in a neighborhood of each end of M. E being an analytic gradient field (except at the charges), this implies that the ω-limit of O must be a critical point [34]. The same argument shows that its α-limit must be either an end or another critical point.

To prove that U is also contained in $\mathcal{F}$, let O be an orbit in U. Its α-limit cannot be a charge, since a charge is a repeller, so it must be either an end or another critical point. In any case, O is not contained in any basin of attraction, so $O \subset \mathcal{F}$. □

Remark 2. Theorem 1 suggests that it can be frequently convenient to think of the boundary as if it were composed of two (not necessarily disjoint) closed sets of different nature, as we shall now outline. Since $M - \mathcal{F}$ is homeomorphic to N disjoint copies of $\mathbb{R}^n$, one can patch these copies together to obtain a disc and find a closed subset $\mathcal{F}_t \subset \mathcal{F}$ so that $M - \mathcal{F}_t$ is homeomorphic to $\mathbb{R}^n$. $\mathcal{F}_t$ then encloses the topological structure of M. The set $\mathcal{F}_s = \overline{\mathcal{F} - \mathcal{F}_t}$ now takes into account the fact that N charges are present, separating the n-disc $M - \mathcal{F}_t$ into N disjoint basins. When $N = 1$, one can consistently take $\mathcal{F}_s = \emptyset$, and when $M \cong \mathbb{R}^n$ one can consider $\mathcal{F}_t = \emptyset$. The decomposition $\mathcal{F} = \mathcal{F}_t \cup \mathcal{F}_s$ is generally not unique.

Theorem 1 also shows that the boundary is composed of electric lines joining an end with a critical point, or connecting two critical points (saddle connection). Example 2 in Section 4 shows that saddle connections can actually appear, even in the simple case of just one charge on a surface.

One should observe that both $\mathcal{F}$ and $\hat{\mathcal{F}}$ can possess rather bad local behavior. However, their structure cannot be extremely pathological. In the following proposition we prove that they cannot be the boundary of Wada basins, contrary to what happens in many other physically relevant dynamical systems; cf. Refs. [26,47] and references therein. In particular, this implies that the points of $\mathcal{F}$ (or $\hat{\mathcal{F}}$) which separate more that two attracting basins constitute a nowhere dense subset.

Proposition 3. *Neither $\mathcal{F}$ nor $\hat{\mathcal{F}}$ possesses the property of Wada.*

Proof. We prove the statement only for $\mathcal{F}$, since for $\hat{\mathcal{F}}$ the proof is completely analogous. Let us suppose that a connected component $\mathcal{F}_0$ of $\mathcal{F}$ possesses the property of Wada. Then it is an indecomposable continuum [26], and hence $\mathcal{F}_0$ is not locally connected at any point. However, $\mathcal{F}_0$ must contain the stable component of a critical point of V, and this stable component is arc-connected, contradicting the fact that $\mathcal{F}_0$ is an indecomposable continuum. □

Since removing the boundary $\mathcal{F}$ from M simply yields N disjoint copies of $\mathbb{R}^n$, one should expect to recover certain homological and homotopical information about M by analyzing the topological structure of $\mathcal{F}$. In the following theorem we show how this goal can be achieved. As a by-product, we will obtain additional results which complement Theorem 1 and Proposition 3 by characterizing the boundary from the viewpoint of shape theory [5,10]. One should observe that the (possibly) bad local properties of the compactified boundary can prevent $\hat{\mathcal{F}}$ from being homeomorphic to a simplicial complex, so Čech homology must be used instead of singular homology. For the same reason, it is preferable to use the coarser notion of shape groups rather than the homotopy groups of $\hat{\mathcal{F}}$ to obtain information about the topology of $\hat{M}$. One should recall [5,10] that the singular homology (resp. homotopy) groups and Čech homology (resp. shape) groups are isomorphic for ANRs, e.g. topological manifolds.

Theorem 2. *For each $k < n - 1$, the k-th homotopy group $\pi_k(\hat{M})$ (resp. homology group $H_k(\hat{M})$) of the compactified space is isomorphic to the k-th shape group $\check{\pi}_k(\hat{\mathcal{F}})$ (resp. Čech homology group $\check{H}_k(\hat{\mathcal{F}})$) of the compactified basin boundary. Furthermore, there exists a monomorphism $\pi_{n-1}(\hat{M}) \to \check{\pi}_{n-1}(\hat{\mathcal{F}})$, and the groups $H_{n-1}(\hat{M}) \oplus \mathbb{Z}^{N-1}$ and $\check{H}_{n-1}(\hat{\mathcal{F}})$ are isomorphic.*

Proof. Let $h : M \to \mathbb{R}$ be any smooth function, positive in $M - \bigcup p_i$, vanishing at each singularity p_i as r^n (r being the geodesic distance to p_i), and such that $h|E|$ tends to zero at each end. Consider the complete smooth vector field on M

$$Y = hE,$$

and let ϕ_t be its flow. By construction, ϕ_t naturally gives rise to a differentiable flow in $\hat{M} - \mathcal{E}(M)$. Since Y vanishes on $\mathcal{E}(M)$, one can extend it to a continuous flow $\tilde{\phi}_t$ on $\hat{M}$ by setting $\tilde{\phi}_t|_{\mathcal{E}(M)} = \mathrm{id}_{\mathcal{E}(M)}$.

Consider N n-discs $B_i \subset \hat{M} - \hat{\mathcal{F}}$ ($i = 1, \dots, N$) such that $p_i \in B_i$ for each i, and let $S = \bigcup B_i$ be their union. For each $j \in \mathbb{N}$, let us define the open set $S_j = \tilde{\phi}_{-j}(S)$, which is obviously homeomorphic to the disjoint union of N n-discs, and its complement $F_j = \hat{M} - S_j$. Since a closed disc with an interior point removed deform retracts onto its boundary, there exists a retraction $R_j : \hat{M} - \bigcup p_i \to F_j$ homotopic (when composed with the inclusion $F_j \to \hat{M} - \bigcup p_i$) to the identity map.

Since $B_i \subset D_i$ and p_i is a global attractor in D_i, it follows that $S_j \subset S_{j+1}$ and $\bigcup S_j = D$, so $\bigcap F_j = \hat{\mathcal{F}}$. Furthermore, F_j deform retracts onto F_{j+1} by construction. Under these conditions a theorem of Borsuk's [4] ensures that $\hat{\mathcal{F}}$ is a FANR, and the trivial homotopy of R_j shows that the fundamental sequence $\underline{R} = \{R_j, \hat{M} - \bigcup p_i \to \hat{\mathcal{F}}\}_{\hat{M}}$ is a strong fundamental deformation retraction [5].

Let us now concentrate on the homotopy groups of $\hat{M}$ and (Borsuk's) shape groups of $\hat{\mathcal{F}}$. Let x_0 and s_0 be arbitrary points of $\hat{\mathcal{F}}$ and S^k, respectively, and let us consider maps $\gamma : (S^k, s_0) \to (\hat{M}, x_0)$ defining elements $[\gamma] \in \pi_k(\hat{M}, x_0)$. For all $k < n$, $\gamma(S^k)$ has empty interior, so one can assume without loss of generality that $p_i \notin \gamma(S^k)$ $(i = 1, \ldots, N)$. Now one can set $\gamma_j = R_j \circ \gamma$ and consider the approximative map $\underline{\gamma} = \{\gamma_j, (S^k, s_0) \to (\hat{\mathcal{F}}, x_0)\}_{\hat{M}}$. The maps $[\gamma_j] \mapsto [\gamma]$ clearly extend to monomorphisms $R_j^{\#} : \pi_k(F_j, x_0) \to \pi_k(\hat{M}, x_0)$, and hence yield a monomorphism $\underline{R}^{\#} : \check{\pi}_k(\hat{\mathcal{F}}, x_0) \to \check{\pi}_k(\hat{M}, x_0)$. Note that $\check{\pi}_k(\hat{M}, x_0) \approx \pi_k(\hat{M}, x_0)$ since $\hat{M}$ is an ANR, and that $\pi_k(\hat{M}, x_0) \approx \pi_k(\hat{M}, x_1)$ for every $x_0, x_1 \in \hat{M}$.

Let $i_j : F_j \to \hat{M}$ be the inclusion map. To prove that the kernel of $R_j^{\#}$ is zero when $k \le n-2$ for all j, and therefore $R_j^{\#}$ and $\underline{R}^{\#}$ are isomorphisms, one should start by observing that $R_j \circ i_j \circ \beta = \beta$ for all $\beta : S^k \to F_j$. Now let $\beta : S^k \to F_j$ belong to the kernel of $R_j^{\#}$, so that $\gamma = i_j \circ \beta$ is null homotopic in $\hat{M}$. Therefore, γ can be extended [48] to a map $\tilde{\gamma} : B^{k+1} \to \hat{M}$, B^{k+1} standing for the $(k+1)$-disc. Thus there exists an extension $\tilde{\beta} = R_j \circ \tilde{\gamma} : B^{k+1} \to \hat{\mathcal{F}}$ of β, which implies that β is also null homotopic in F_j.

Let us now prove the statement on the homology groups. First, let B_N denote the disjoint union of N n-discs and $\dot{B}_N$ the disjoint union of N punctured n-discs. Since $\hat{\mathcal{F}}$ is a strong fundamental deformation retract of $\hat{M} - \bigcup p_i$, they have the same shape [5], and thus $\check{H}_k(\hat{M}, \hat{\mathcal{F}}) \approx \check{H}_k(\hat{M}, \hat{M} - \bigcup p_i)$ for all k. By the excision axiom, $\check{H}(\hat{M}, \hat{\mathcal{F}})$ is then isomorphic to $\check{H}_k(\hat{M} - \hat{\mathcal{F}}, \hat{M} - (\hat{\mathcal{F}} \cup \bigcup p_i))$. Therefore from Proposition 2 it follows that

$$\check{H}_k(\hat{M}, \hat{\mathcal{F}}) \approx \check{H}_k(B_N, \dot{B}_N) \approx \begin{cases} 0 & \text{if } k < n \\ \mathbb{Z}^N & \text{if } k = n. \end{cases}$$

Hence the exact sequence $0 \to \check{H}_k(\hat{\mathcal{F}}) \to \check{H}_k(\hat{M}) \to 0$ shows that $\check{H}_k(\hat{\mathcal{F}}) \approx \check{H}_k(\hat{M})$ for $k < n-1$, whereas the exact sequence $0 \to \mathbb{Z} \to \mathbb{Z}^N \to \check{H}_{n-1}(\hat{\mathcal{F}}) \to \check{H}_{n-1}(\hat{M}) \to 0$ proves that $\check{H}_{n-1}(\hat{\mathcal{F}}) \approx \check{H}_{n-1}(\hat{M}) \oplus \mathbb{Z}^{N-1}$. Since $\hat{M}$ is an ANR, $\check{H}_k(\hat{M}) \approx H_k(\hat{M})$, and the claim follows. □

Corollary 1. *$\mathcal{F}$ is either empty or non-compact. Furthermore, $\hat{\mathcal{F}}$ is a connected FANR, and has the shape of a finite polyhedron in standard position.*

Proof. In the proof of Theorem 2 it was proved that $\hat{\mathcal{F}}$ is a FANR. Connectedness stems from the fact that $\mathcal{F} = \bigcap F_j$ and each F_j is connected. By Ref. [19], it also implies that $\hat{\mathcal{F}}$ has the shape of a finite polyhedron in standard position. Furthermore, as $\hat{\mathcal{F}} = \mathcal{F} \cup \mathcal{E}(M)$ is connected, then either $\mathcal{F}$ is empty or $\mathcal{E}(M) \cap \overline{\mathcal{F}} \neq \emptyset$ in the compactified manifold. □

As an interesting physical consequence, one should note that in principle an experimentalist could use this theorem to gain some insight into the topological structure of the physical space M by detecting the points at which the electric field generated by a point charge vanishes and following the directions in which the electric lines enter the critical points.

4. Electric fields on surfaces

In this section we will study the topology of the basin boundary on surfaces, where the results of the previous section can be strengthened. A useful elementary property in dimension 2 is the conformal invariance of Li and Tam's fundamental solutions. In this section, M will always denote a 2-manifold.

Lemma 1. *If (M, g) and $(M, \tilde{g})$ are conformally isometric, they admit the same Li–Tam fundamental solution.*

Proof. Let $\tilde{g} = \lambda g$, where $\lambda : M \to \mathbb{R}^+$. Then it is well known that the Laplacian and delta distribution in (M, g) and $(M, \tilde{g})$ are related by $\widetilde{\Delta} = \lambda^{-1}\Delta$ and $\tilde{\delta}_p = \lambda^{-1}\delta_p$. Therefore the equation $-\Delta V_p = \delta_p$ is conformally invariant, and the lemma follows. □

It is easy to observe that a closed n-manifold does not admit any fundamental solutions, since they would be non-constant and would necessarily attain their minimum in $M - p$, contradicting their harmonicity in $M - p$. Lemma 1 provides a simple proof of the following related fact, which had already been approached [54] using the theory of holomorphic functions.

Proposition 4. *The electric field generated by the charge configuration $\mathcal{C}$ on a closed 2-manifold exists if and only if $\sum q_i = 0$.*

Proof. The "only if" part is elementary, since

$$-\sum_{i=1}^{N} q_i = \int_M \Delta V \mathrm{d}x = \int_{\partial M} \frac{\partial V}{\partial n} \mathrm{d}\sigma = 0$$

as M is closed.

Let us now concentrate on the "if" part. Without loss of generality, one can restrict to the case $\mathcal{C} = \{(1, p), (-1, p')\}$, since the general case follows from linear superposition by decomposing the original configuration into neutral pairs. Let $\tilde{M} = M - p'$, which is not a complete manifold with the induced metric. From Ref. [38] it follows that there exists a smooth conformal factor $\lambda : M \to \mathbb{R}^+$ such that $(\tilde{M}, \lambda g)$ is complete. Let $\tilde{V}$ be a Li–Tam fundamental solution, which must tend to $-\infty$ at p' since it has a well defined limit at the only end p' of $\tilde{M}$, and otherwise it would be extendable to a fundamental solution on a closed manifold. It can however be extended to a singular function $V : M \to \overline{\mathbb{R}}$ so that $V \in C^\omega(M - p - p')$. By Ref. [13], p' is a Newtonian singularity of V, which must satisfy $-\Delta V = \delta_p + q\delta_{p'}$. As the sum of the charges must vanish, $q = -1$. □

Remark 3. The conformal factor λ must tend to $+\infty$ at p'. This follows from a theorem of Gordon [16], asserting that a Riemannian manifold is complete if and only if there exists a proper function f whose gradient is bounded in norm. Thus, if $\tilde{\nabla}$ and $|\cdot|_{\tilde{M}}$ denote the gradient and norm in $(\tilde{M}, \lambda g)$, there exists a proper function f and a constant c such that $c \geq |\tilde{\nabla} f|^2_{\tilde{M}} = \lambda^{-1}|\nabla f|^2$ in $M - p'$. Since $(\tilde{M}, g)$ is not complete, there does not exist a constant c' satisfying $|\nabla f|^2 = \lambda|\tilde{\nabla} f|^2_{\tilde{M}} \leq c'$, so one must have $\lim_{x\to p'} \lambda(x) = +\infty$.

Note that the proposition above does not claim the electric field to be independent of the order in which the neutral pairs of charges are taken. One could also be tempted to consider that negative charges are equivalent to "holes" in the manifold. Proposition 4 shows that negative charges in a closed manifold can indeed be identified with holes, in a certain sense. In open manifolds, however, this identification is generally not possible, as the following example shows.

Example 1. Let us consider the electric field generated by two point charges $\mathcal{C} = \{(-1, x_-), (1, x_+)\}$, where $x_- = (0, 0)$ and $x_+ = (-1, 0)$, in the Euclidean plane. The standard Li–Tam potential is given by

$$V(x) = \frac{1}{2\pi}(\log|x| - \log|x - x_+|). \tag{6}$$

Let $\Phi : \mathbb{R}^2 - x_- \to \mathbb{R} \times S^1$ be the diffeomorphism which maps the punctured plane into the cylinder by defining $z = \log|x|$ and taking $\theta \in S^1$ as the polar angle determined by x. The induced metric is $\mathrm{d}s^2 = \mathrm{e}^{2z}(\mathrm{d}z^2 + \mathrm{d}\theta^2)$, and is conformally equivalent to the flat metric. By Lemma 1 it follows that the induced potential $\tilde{V} = \Phi_* V$ is a fundamental solution in the flat cylinder, but it is not of Li–Tam type. To see this, note that $\lim_{z\to-\infty} \tilde{V}(z, \theta) = -\infty$ whereas $\lim_{z\to+\infty} \tilde{V}(z, \theta) = 0$, so that one end is parabolic the other one is hyperbolic. However, a criterion appearing in Ref. [31] shows that both ends must be parabolic for any Li–Tam fundamental solution. Hence, negative charges and holes cannot be generally regarded as equivalent concepts.

In the following proposition we concentrate on the local and global topological structure of the basin boundary, which can be described more thoroughly for surfaces than it can be in arbitrary dimension; cf. Section 3. In the rest of this section, M is an open surface, which can be topologically characterized by the number of handles g and of holes h according to Richards' theorem [43]. Furthermore, $\mathcal{C}$ again denotes a configuration of negative point charges. One should also observe that the critical set of V in a 2-manifold must be composed of isolated points.

Proposition 5. *Let x be a critical point of V. Then x is a topological saddle with $2m$ half-branches, where $m \geq 2$ is the degree of the lowest homogeneous term in the Taylor expansion of V near x.*

Proof. Let (r, θ) be polar Riemann coordinates centered at x. One can make the expansion $V(r, \theta) = cr^m f(\theta) + O(r^{m+1})$, where $m \geq 2$, as x is a critical point of V. By harmonicity, one can set $f(\theta) = \cos m\theta$ without loss of generality. The equation $\dot{x} = -\nabla V$ can be blown up into

$$\dot{r} = mr \cos m\theta + O(r^2), \tag{7}$$

$$\dot{\theta} = -m \sin m\theta + O(r). \tag{8}$$

Then the blown-up critical points are given by $r = 0$, $\theta_k = k\pi/m$, where $k = 1, \dots, 2m$. The linearization of the blown-up field at the point $(0, \theta_k)$ is given by $\mathrm{diag}(\pm m, \mp m)$, and therefore they are hyperbolic saddles. Thus the original field possesses a topological saddle at x, whose $2m$ half-branches are tangent at x with angle θ_k. □

Corollary 2. *In a sufficiently small neighborhood U of the critical point x, $\mathcal{F} \cap U$ is composed of m half-branches. In particular, $\mathcal{F}$ has no endpoints and is triangulable.*

Proof. By Theorem 1, $\mathcal{F}$ is composed of the critical points of V and their stable components. From the change of sign of $\dot{r}$ in Eq. (7) at the blown-up critical points, it follows that the stable and unstable half-branches alternate, and hence one has m stable and m unstable half-branches. If $\mathcal{F}$ had an endpoint, it would be a critical point x of V. However, $m \geq 2$ stable half-branches fall into x, so x cannot be an endpoint. Furthermore, since $\mathcal{F}$ has pure dimension 1 (provided it is non-empty) and each half-branch is an electric line and hence a differentiable submanifold, $\mathcal{F}$ is triangulable. □

From Corollary 2 the configuration $\mathcal{C}$ in M naturally yields a decomposition of the compactified manifold $\hat{M}$ into 0-cells x_i, h_j, 1-cells γ_i and 2-cells D_i. Here and in what follows, x_i represent the critical points of V, and h_j the ends (holes) of the manifold. Furthermore, γ_i denote the stable electric lines associated with the critical points, and D_i is the basin of the i-th charge. The boundary of a k-cell is composed of cells of dimension up to $k - 1$.

Remark 4. In dimension 2 an enlightening visual picture of $\mathcal{F}$ can be obtained by representing $\hat{M}$ as a $2g$-gon with identified faces. Let us consider the simplest case, $N = 1$. As a consequence of Theorem 2, $\mathcal{F}$ must contain $2g$ homotopically independent loops, which can be realized as the border of the $2g$-gon. Furthermore, $\mathcal{F}$ cannot contain any other loop, since it would separate regions without charge and thus contradict Proposition 1. These $2g$ loops constitute a closed subset of $\mathcal{F}$ which we could call a *loop boundary*. The rest of the boundary must be constituted by curves which do not destroy the contractibility of the interior of the $2g$-gon when removed from it. Thus $\mathcal{F}$ is made of $2g$ independent loops, and of non-periodic curves with an endpoint lying at the border of the $2g$-gon and the other one being a compactified hole of M.

We shall now study upper and lower bounds for the number of critical points of V which rest upon the local analysis performed in Proposition 5. We shall denote by N_Z the number of critical points x_i of V, and by m_i the number of stable half-branches at x_i. We also define the branch number $N_B = \sum_i m_i$.

Theorem 3. *If N_Z is finite, the following bounds for the number of critical points and the branch number hold:*

$$\max\{1 - \delta_{N,1}\delta_{g,0}\delta_{h,1}, 2g - h - N + 2\} \leq N_Z \leq 2g + h + N - 2, \quad 4g \leq N_B \leq 2(2g + h + N - 2),$$

where $\delta_{i,j}$ stands for the Kronecker delta. Furthermore, the upper bound for N_Z is attained if and only if all the critical points are hyperbolic.

Proof. Since the index of a saddle with $2m$ half-branches is $1 - m$ [40], Proposition 5 implies that the index of x_i is upper bounded by -1, and equals -1 if and only if x_i is hyperbolic.

As a consequence of Richards' theorem [43], the compactified manifold $\hat{M}$ can be endowed with a differentiable structure, which is unique as $\hat{M}$ is a 2-manifold [21]. It is therefore easy to regularize the induced electric field on $\hat{M}$ so that the charges p_i and the holes h_i become critical points of the regularized smooth vector field $\hat{E}$. Now one can apply Hopf's index theorem [40] to obtain

$$\chi(\hat{M}) = \sum_{i=1}^{N} \mathrm{ind}_{\hat{E}}(p_i) + \sum_{i=1}^{h} \mathrm{ind}_{\hat{E}}(h_i) + \sum_{i=1}^{N_Z} \mathrm{ind}_{\hat{E}}(x_i) = N + h + \sum_{i=1}^{N_Z} \mathrm{ind}_{\hat{E}}(x_i),$$

where $\chi(\hat{M}) = 2 - 2g$ is the Euler characteristic of $\hat{M}$, we denote the points in M and their projection in $\hat{M}$ by the same symbol, and we have used that the index of the attractors p_i and the repellers h_i is 1. Since $\mathrm{ind}_{\hat{E}}(x_i) \leq -1$, the upper bound for N_Z follows.

The lower bound for N_B can be obtained by realizing that $\mathcal{F}$ contains at least the $2g$ loops of the loop boundary, and therefore there are at least $4g$ stable half-branches. The upper bound for N_B can be derived from the one for N_Z and the equation above:

$$2g + h + N - 2 = -\sum_{i=1}^{N_Z} \mathrm{ind}_{\hat{E}}(x_i) = \sum_{i=1}^{N_Z} (m_i - 1) = N_B - N_Z.$$

Finally, this equation and the lower bound for N_B show that

$$N_Z = 2 - N - 2g - h + N_B \geq 2g - h - N + 2.$$

Furthermore, $N_Z > 0$ whenever the boundary is non-empty, as will be the case when $M \ncong \mathbb{R}^2$ ($g > 0$ or $h > 1$) or $N > 1$. □

Corollary 3. *If N_Z is finite, the potential generated by one charge in the plane with g handles has exactly $2g$ critical points, which are all hyperbolic. Moreover, $\hat{\mathcal{F}}$ coincides with the loop boundary.*

Proof. By Theorem 3, V has exactly $2g$ critical points and they are hyperbolic. To prove that $\hat{\mathcal{F}}$ coincides with the loop boundary, by Remark 4 it is enough to show that the compactified hole h_1 lies on the loop boundary. Let us assume that $g > 0$, since the case $g = 0$ is trivial, and suppose that h_1 lies outside the loop boundary. Remark 4 shows that one can continue the electric line starting at h_1 until it reaches a critical point x_1 in the loop boundary, possibly after passing over other critical points of V. As each critical point has two stable half-branches, only one curve of $\hat{\mathcal{F}}$ can fall into h_1, and there are a finite number of critical points, the other stable half-branch of x_1 can be continued through other stable half-branches to obtain a cycle γ contained in the loop boundary. Since γ is made of stable components,

$$0 < \int_\gamma \mathrm{d}V = 0,$$

and hence we reach a contradiction. □

One should note that, as Remark 2 would suggest, the upper bound for N_Z, $2g + h + N - 2$, is the sum of the first Betti number $b_1(M) = 2g + h - 1$, which takes into account the non-trivial topology of M, and $N - 1$, corresponding to the maximum number of components which would separate N 2-discs in the plane. In particular, the upper bound would be saturated by a Morse–Smale electric field. An interesting open question is that of proving (or disproving) that the electric fields on surfaces are generically Morse–Smale. When M is diffeomorphic to $\mathbb{R}^2$, (M, g) is conformally isometric [49] to either the Euclidean plane or the hyperbolic 2-space, and Lemma 1 can be used to prove that the electric field is generally Morse–Smale in this case. Should this property hold for an arbitrary surface, then the electric field, and hence the boundary, would be structurally stable in the generic case [27], and the upper bound for N_Z would be sharp.

Now we present an example showing that the electric field need not be Morse–Smale even when $N = 1$, as it can have saddle connections.

Example 2. Let T be the 2-torus $\{(z, w) \in \mathbb{C}^2 : |z| = |w| = 1\}$, with external equator $\{z = 1\}$ and internal equator $\{z = -1\}$, and consider the collinear points $p = (1, 1)$, $x_1 = (-1, 1)$, $x_2 = (-1, -1)$ and $h = (1, -1)$. Let (M, g) be the surface $T - h$, endowed with a complete conformally flat metric, and consider a negative charge at p. Let us define the diffeomorphisms of M given by $e(z, w) = (z, w)$, $a_1(z, w) = (z^{-1}, w)$, $a_2(z, w) = (z, w^{-1})$, which are obviously isometries of M when endowed with the (incomplete) flat metric. Furthermore, the groups $G_1 = \{e, a_1\}$ and $G_2 = \{e, a_2\}$ are symmetries of the charge configuration. By Lemma 1 and Corollary 5 in Section 6, the curves $\{z = \pm 1\}$ and $\{w = \pm 1\}$ are invariant under the electric field, and therefore it can be easily seen using Theorem 3 that the points x_1 and x_2 are the only critical points, which are hyperbolic, and that the invariant set $\{z = -1\}$ constitutes a saddle connection.

5. Geodesic behavior

When $N = 1$, the results of Section 3 show that the topological properties of the boundary $\mathcal{F}$ closely resemble those of the cut locus of a Riemannian manifold. This is particularly remarkable since the cut locus is a subanalytic set [7], whereas generally the basin boundary is not even known to be triangulable. We shall therefore devote this section to analyzing the relationship between the geodesics and the electric lines on the one hand, and the cut locus and the boundary on the other. We always assume that the charge configuration is $\mathcal{C} = \{(-1, p)\}$ $(N = 1)$.

First let us introduce some standard notation [41]. By $C(p) \subset M$ and $\Sigma_p \subset T_pM$ we shall denote the cut locus and the segment domain at p, respectively, and $\exp_p : T_pM \to M$ will stand for the exponential map at p. It is well known that $C(p)$ has dimension at most $n - 1$, and that $\exp_p$ diffeomorphically maps the interior of the closed n-disc Σ_p into $M - C(p)$.

Proposition 6. *There exists an analytic diffeomorphism $D \to M - C(p)$ mapping the electric lines into the geodesics starting at p.*

Proof. By Proposition 2, one can take an analytic global chart (x^i) in D and define the complete vector field X as in Eq. (4). Let ϕ_t be its flow, which is analytic. In Proposition 1 it has been proved that there exist a neighborhood U of p and an analytic diffeomorphism $\Psi : U \to B^n$ which maps the electric lines into straight lines passing through the origin. Here $B^n \subset \mathbb{R}^n$ denotes the unit ball and one can assume that U is saturated by V, so that $\partial U = V^{-1}(c)$.

Consider a singular foliation of U defined by λ, where $\lambda : U \to [0, +\infty)$ is the onto analytic function defined as $\lambda(x) = (\mathrm{e}^{-V(x)} - \mathrm{e}^{-c})^{-1}$. Clearly the leaves of the foliation are those of V, i.e., topological spheres centered at p, and λ is a Lyapunov function of the vector field X.

Let $\Phi : U \to D$ be the analytic map $\Phi(x) = \phi_{-\lambda(x)}x$. Obviously Φ leaves the electric lines invariant, and is bijective and bicontinuous since λ decreases along the orbits of X. Its inverse, which has the form $\Phi^{-1}(x) = \phi_{g(x)}x$, where $g(x) = \lambda(y)$ and $x = \Phi(y)$, is also analytic, and Φ defines an analytic diffeomorphism.

Now one can construct an analytic diffeomorphism $\psi : B^n \to \operatorname{int} \Sigma_p$ which preserves the straight lines passing through the origin simply by dragging along the radial directions. Since the restriction $\exp_p : \operatorname{int} \Sigma_p \to M - C(p)$ is also an analytic diffeomorphism, $\exp_p \circ \psi \circ \Psi \circ \Phi^{-1} : D \to M - C(p)$ provides the desired diffeomorphism. □

Remark 5. Generally, this diffeomorphism cannot be extended to a homeomorphism $M \to M$ mapping the electric lines into the geodesics globally. In fact, recall that the cut locus of a surface can be homeomorphic to a half-line in a neighborhood of one of its points (e.g., in the paraboloid, when p is not the vertex), while in Corollary 2 we proved that the basin boundary of a surface cannot have any endpoints. Therefore, $C(p)$ and $\mathcal{F}$ are not generally homeomorphic via a homeomorphism $M \to M$.

Example 3. Generally speaking, the basin boundary is not contained in $C(p)$ either. To see this, let us consider the cylinder $S^1 \times \mathbb{R}$ with the metric given in local coordinates by $\mathrm{d}s^2 = f(\theta)(\mathrm{d}\theta^2 + \mathrm{d}z^2)$, where $-\infty < z < +\infty$, $-\pi < \theta < \pi$, and f is positive and 2π-periodic. In these coordinates, one can assume that the position p of the charge is $z = 0, \theta = 0$.

The geodesic equation reads

$$\frac{\mathrm{d}}{\mathrm{d}t}(f(\theta)\dot{z}) = 0,$$

$$2\frac{\mathrm{d}}{\mathrm{d}t}\left(f(\theta)\dot{\theta}\right) - \dot{z}^2 f'(\theta) = 0,$$

where t denotes the arc length. The geodesics contained in the invariant set $\{z = 0\}$ starting at p can be easily obtained by the quadrature

$$\int_0^{\theta} f(\overline{\theta})\mathrm{d}\overline{\theta} = ct,$$

c being a constant. Thus the intersection $C(p) \cap \{z = 0\}$ is given by $(\theta_0, 0)$, θ_0 being the solution, unique modulo 2π, to

$$\int_0^{\theta_0} f(\overline{\theta})\mathrm{d}\overline{\theta} = \int_0^{2\pi-\theta_0} f(\overline{\theta})\mathrm{d}\overline{\theta}.$$

Example 6 in Section 6 and Lemma 1 show that in the conformally flat cylinder the basin boundary is given by $\mathcal{F} = \{\theta = \pi\}$. Since $\theta_0 \neq \pi$ generally, this establishes that $\mathcal{F} \not\subset C(p)$.

Theorem 2 and Remark 5 show that the electric lines arise as curves of a new kind on a Riemannian manifold which do not generally coincide with geodesics and of intrinsic geometrical and topological interest. They define in a natural way a decomposition of the manifold into the basin boundary $\mathcal{F}$ (of dimension at most $n-1$ and containing most of the homotopical and homological information of M), and Nn-cells bounded by $\mathcal{F}$. This decomposition is not generally homeomorphic to the one obtained from the cut locus [55,42] and, unlike a vector field given by geodesics emanating from one point [36], it induces non-trivial dynamics on the basin boundary. It is remarkable that this decomposition, which is standard in the sense of Doyle and Hocking [12], is given by a simple vector field whose origin is rooted in classical physics.

In the following theorem we characterize in which situations the electric lines are geodesics, and thus the two kinds of lines coincide. Of course, analyticity implies that the electric lines are globally geodesic whenever they are locally, and the contact order of geodesics and electric lines at p is at least 2; cf. Eqs. (2) and (5). Recall that a space (M, g) is harmonic [3] with respect to p if the volume density function in normal Riemann coordinates centered at p, which we denote by $\sqrt{G}$, only depends on the geodesic distance to p.

Theorem 4. *The electric lines emanating from p are geodesics if and only if the cut locus $C(p)$ is empty and the space is harmonic with respect to p.*

Proof. We begin with the "if" part. Let (x^i) be normal Riemann coordinates (NRC) centered at p, which are globally defined since $C(p) = \emptyset$, and let $(r, \theta) \in \mathbb{R}^+ \times S^{n-1}$ be polar Riemann coordinates (PRC), i.e., the spherical coordinates associated with (x^i). In PRC, the metric reads $\mathrm{d}s^2 = \mathrm{d}r^2 + g_{ij}(r, \theta)\mathrm{d}\theta^i \mathrm{d}\theta^j$. The determinant of the metric in these coordinates is $\tilde{G} = r^{2n-2} G(r)\sigma(\theta)$, where $\sqrt{G(r)}$ is the volume density function in NRC and $\sqrt{\sigma(\theta)}$ is the volume density function of the round unit $(n-1)$-sphere in spherical coordinates. It can be readily verified that in this case the potential

$$V = c_n \int \frac{\mathrm{d}r}{\sqrt{G(r)} r^{n-1}} \tag{9}$$

is a Li–Tam solution to the equation $\Delta V = \delta_p$, proving the claim.

Let us now address the converse implication. One can prove that $C(p)$ is contained in $\mathcal{F}$, since $E(x) = 0$ for all $x \in C(p)$. To see this, note that at each x in $C(p)$, either two geodesics intersect or the derivative $D\exp_p$ vanishes. In the first case, E must vanish because two different orbits cannot intersect. In the second case, let us suppose that x is not a critical point of E. Then there exists a smooth reparametrization of the electric line γ in a neighborhood of x so that one has $\dot{\gamma} = D\exp_p \partial_r = 0$, contradicting $E(x) \neq 0$.

If $C(p) \neq \emptyset$, $C(p)$ is strictly contained in $\mathcal{F}$, since by Theorem 1 $\mathcal{F}$ also contains the stable components of the critical set of E, which must be non-empty because E is divergence-free. As $M - C(p)$ deform retracts to p by dragging along the geodesics, which are also electric lines by hypothesis, the ω-limit of the non-empty set $\mathcal{F} - C(p)$ is p, contradicting its definition. Hence $C(p) = \emptyset$.

To prove that G only depends on r, let us take Riemann coordinates, which are globally defined. By hypothesis, the electric field must have the form $E = f(r, \theta)\partial_r$. As it is divergence-free in $M - p$,

$$\mathrm{div} E = \frac{1}{\sqrt{\tilde{G}}} \frac{\partial}{\partial r} \left[\sqrt{\tilde{G}} f(r, \theta)\right] = 0,$$

so that $f(r, \theta) = \tilde{G}^{-1/2} f_1(\theta)$. Since E is irrotational,

$$\mathrm{d}E^\flat = \frac{\partial}{\partial \theta^i} \left[\tilde{G}^{-1/2} f_1(\theta)\right] \mathrm{d}\theta^i \wedge \mathrm{d}r = 0,$$

and hence $\tilde{G} = f_1(\theta)^2 f_2(r)$ and $G = g_1(\theta) g_2(r)$. As $G = 1$ at p $(x^i = 0)$, one must have $1 = g_1(\theta) g_2(0)$, which implies that $g_1(\theta)$ is a constant and proves the assertion. □

Eq. (9) had already appeared in the literature within the context of *local* fundamental solutions in harmonic spaces [46], i.e., manifolds which are harmonic with respect to every point. However, the full characterization given above, which includes spaces in which the only electric lines assumed to be geodesic are those which emanate from a given point p, and the relationship between the cut locus and the existence of *global* radial fundamental solutions seems to have escaped notice. Let us now discuss a couple of relevant examples.

Example 4. Important examples of harmonic spaces with empty cut locus are the non-compact two-point homogeneous spaces [3] $\mathbb{R}^n$ and $\mathcal{H}^j(\mathbb{K})$, where the field $\mathbb{K}$ is either the reals $\mathbb{R}$, the complex numbers $\mathbb{C}$, the quaternions $\mathbb{Q}$ or the octonions $\mathbb{O}$. Theorem 4 implies that the electric field in the hyperbolic space $\mathcal{H}^j(\mathbb{K})$ is given by

$$E = -\frac{c_{j\nu}}{\sinh^{j\nu-1} r \cosh^{\nu-1} r}\partial_r,$$

where ν is the real dimension of the field $\mathbb{K}$. Note that in these spaces the electric lines generated by a point charge are always geodesics, just as in Euclidean space. In particular, the classification of harmonic manifolds up to dimension 4 [52] implies that the only spaces of dimension ≤ 4 possessing this property are $\mathbb{R}^l$, $\mathbb{H}^l \equiv \mathcal{H}^l(\mathbb{R})$, $\mathcal{H}^k(\mathbb{C})$ and $\mathcal{H}^1(\mathbb{Q})$, with $1 \leq l \leq 4$ and $k = 1, 2$. Obviously there are harmonic spaces whose cut locus is non-empty and therefore their electric lines are not (locally) geodesic, e.g., the flat cylinder; cf. Example 6.

Example 5. Rotationally symmetric spaces with respect to p diffeomorphic to $\mathbb{R}^n$ (e.g., the paraboloid, if p is the vertex) satisfy the hypotheses of Theorem 4. The metric has the form $\mathrm{d}s^2 = f(R)^2\mathrm{d}R^2 + R^2\mathrm{d}\Omega^2$, where $\mathrm{d}\Omega^2$ is the metric of the round unit $(n-1)$-sphere and one can assume $f(0) = 1$. The electric field is now given by $E = -c_n R^{1-n} f(R)\partial_R$. Cheng and Yau's necessary condition for the existence of a positive Green function [9] is in this case also sufficient, since $\int \mathrm{vol}_p(r)^{-1} r\mathrm{d}r$ behaves at infinity as the potential because $r = \int f(R)\mathrm{d}R$ and $\mathrm{vol}_p(r) = c_n^{-1}\int f(R)R^{n-1}\mathrm{d}R$. Note that it is also true that electric fields on asymptotically flat spaces satisfying the hypotheses of Theorem 4 have Euclidean behavior at infinity.

As is well known in potential theory, the fact that the electric lines generated by a point charge in Euclidean space are straight implies that the electric field generated by a charge distribution is radial if and only if the latter distribution has spherical symmetry. We shall extend this property to arbitrary manifolds, and thus provide another characterization of spaces whose electric lines emanating from p are geodesics.

Let $\rho : M \to \mathbb{R}$ be a piecewise smooth charge distribution which does not vanish identically. The potential generated by ρ will be denoted by V, and the one generated by the negative unit charge at p by V_p. We say that two piecewise smooth functions $f, g : M \to \mathbb{R}$ agree fiberwise if $\mathrm{d}f \wedge \mathrm{d}g = 0$, which implies that they define the same foliation at the points where f and g are regular.

Theorem 5. *The electric lines emanating from p are geodesics if and only if V and V_p agree fiberwise for one, and therefore all, ρ agreeing fiberwise with V_p.*

Proof. First we prove the direct implication. The exponential map at p is globally defined by Theorem 4, and ρ depends only on r since it agrees fiberwise with V_p. Let us define the function $Q(r) = \int_0^r \rho(\bar{r})\bar{r}^{n-1}G(\bar{r})^{1/2}\mathrm{d}\bar{r}$. Then it can be readily verified that the potential

$$V(r) = -\int \frac{Q(r)}{\sqrt{G(r)}r^{n-1}}\mathrm{d}r$$

constitutes a Li–Tam solution to the equation $-\Delta V = \rho$.

Now we prove the converse implication. Let U be a sufficiently small domain in $\mathrm{supp}\rho - p$ where $\mathrm{d}V_p \neq 0$. Then one can write $V = f_1(V_p)$ and $\rho = f_2(V_p)$ in U, and Poisson's equation reads

$$f_2(V_p) = f_1''(V_p)|\nabla V_p|^2,$$

so V_p is a transnormal function [53] in U, and therefore throughout its analyticity domain $M - p$. The equipotential sets $\{V_p = c\}$ near p are topological spheres by Proposition 1, and parallel by the transnormality condition $|\nabla V_p|^2 = f(V_p)$. Since V_p is regular in a punctured neighborhood of p, its focal set is p, and therefore the equipotential sets of V_p near p must be geodesic spheres. This implies that the electric lines are geodesic locally, and hence globally. □

Theorems 4 and 5 show that harmonic spaces (with respect to p) with empty cut locus reproduce the physically most relevant aspects of potential theory in Euclidean spaces. It is remarkable that the appropriate generalization of charge distributions whose field can be computed as if generated by a point charge leads not only to spherically symmetric spaces but also to the wider class of harmonic manifolds with empty cut locus.

6. Symmetries and spaces of constant curvature

In Eq. (9) we obtained a closed expression for the potential assuming that the electric lines were geodesics. Generally such a closed expression cannot be found, and in this case symmetries provide a useful means of extracting geometrical information about the orbits of the electric field.

In the following proposition we prove that there exists a Li–Tam fundamental solution which inherits the isometries of the space. From a physical viewpoint, it is natural to choose this kind of fundamental solution to define the potential function, and therefore we shall always assume that such a choice has been made.

Proposition 7. *Let G be a closed Lie subgroup of isometries of (M, g). Then there exists a Li–Tam fundamental solution $v : M \times M \to \mathbb{R}$ such that $v(ax, ay) = v(x, y)$ for all $a \in G$.*

Proof. Let $v_0 : M \times M \to \mathbb{R}$ be any Li–Tam fundamental solution, and let us define an action of G on $M \times M$ given by $a \cdot (x, y) = (x, ay)$. Let $U \subset M$ be an open n-disc on which G acts freely, and define $W = M \times U$. Since G is closed, the orbit space W/G can be realized [39] as an embedded submanifold of W transverse to the orbits of G. For each $y \in U$ one can find $a_y \in G$ and $\hat{y} \in U/G$ such that $y = a_y\hat{y}$, and such a decomposition is unique. Hence one can define a function $\overline{v} : W \to \mathbb{R}$ as $\overline{v}(x, y) = v_0(a_y^{-1}x, \hat{y})$. This implies that $\overline{v}(ax, ay) = \overline{v}(x, y)$ whenever y and ay belong to U. Furthermore, since the Laplacian commutes with isometries, $\overline{v}$ satisfies that $-\Delta_x\overline{v}(x, y) = \delta_y(x)$ in W. As the solutions of this equation are analytic for $x \neq y$, there exists an extension $v : M \times M \to \mathbb{R}$ of $\overline{v}$, analytic in $\{(x, y) \in M \times M : x \neq y\}$, which by analyticity must be a Li–Tam fundamental solution satisfying $v(ax, ay) = v(x, y)$. □

Corollary 4. *Let ρ be a charge distribution which is invariant under a closed Lie subgroup of isometries G. Then both V and E are also invariant under G.*

Proof. To prove that V is invariant under G, observe that

$$V(ax) = \int_M v(ax, y)\rho(y)\mathrm{d}y = \int_M v(x, y)\rho(ay)\mathrm{d}ay = V(x)$$

by Proposition 7. Furthermore, since a is an isometry, $(a_*\nabla V)(x) = (\nabla V)(ax)$. □

Remark 6. It can be easily seen that these results also hold for finite discrete subgroups of isometries, and when the charge distribution is substituted by a configuration of point charges. It also applies for conformal isometries (i.e., $(a_*g)(x) = cg(ax)$, where g is the metric tensor, $a \in G$, and c is a constant) when one replaces "invariant" by "conformally invariant".

Corollary 5. *Let (M, g) be a Riemannian 2-manifold conformally isometric to $(M, \tilde{g})$, and let G be a subgroup of isometries of $(M, \tilde{g})$ as in Proposition 7 or Remark 6. Let us suppose that G is a symmetry group of a configuration of point charges $\mathcal{C}$. Then G is a symmetry group of V and a generalized symmetry group of E.*

Proof. By Lemma 1 (M, g) and $(M, \tilde{g})$ admit the same Li–Tam fundamental solution $\tilde{v}$, which inherits the isometry subgroup G from $(M, \tilde{g})$. Therefore the potential $V = \sum q_i\tilde{v}(\cdot, p_i)$ is also G-invariant. By the conformal symmetry $\tilde{g} = \lambda g$, one can write $(a_*E)(x) = \lambda(ax)\lambda(x)^{-1}E(x)$ for each $a \in G$, and hence G maps orbits of E into orbits of E. □

It can be readily verified that, conversely, an isometry which leaves the potential invariant is also a symmetry of the charge distribution. Observe that, as a consequence of the Corollary 4 and Remark 6, the boundary $\mathcal{F}$ must be invariant under the closed subgroup G of isometries which preserve the charge configuration. Therefore, $\mathcal{F}$ is saturated by orbits of G.

Corollary 4 provides another method of proving that $\mathcal{F}$ is empty when (M, g) is rotationally symmetric with respect to p and $\mathcal{C} = \{(-1, p)\}$ (note that this can also be proved using Theorem 4). In particular, the electric field does not have any critical points. To prove this result, observe that M must be diffeomorphic to $\mathbb{R}^n$, and that spherical symmetry leaves the charge at p invariant, so by Corollary 4 $\mathcal{F}$ would be composed of $(n-1)$-spheres, which would separate off a closed region without charge, contradicting Proposition 1.

A useful observation concerning homogeneous spaces that stems from Proposition 7 is that in these manifolds it suffices to calculate one potential function V_p to obtain the potential generated by any configuration of point charges $\mathcal{C}$. Actually, since the isometry group is transitive in homogeneous spaces, there exist isometries a_i such that $p = a_i p_i$, and one can express the potential as $V(x) = \sum q_i V_p(a_i x)$. This can be applied, e.g., to hyperbolic space, where the expression for V_p was given in Example 4 in Section 5.

Symmetries of vector fields frequently give rise to first integrals and invariant sets, which in turn can be used sometimes to obtain exact solutions to non-trivial problems by reducing the solution of a simpler problem in higher dimension to an invariant subset; cf. e.g. [25,8]. Nevertheless, this approach does not seem to yield significant results in the study of electric fields on manifolds for the reasons that we shall shortly discuss.

For instance, let us assume that S is an analytic submanifold of M invariant under the electric field E, and let $j : S \to M$ be an embedding. Let us suppose that the first N' charges of a configuration $\mathcal{C} = \{(q_i, p_i)\}_{i=1}^{N}$ lie on S. It is natural to ask whether the induced vector field $\tilde{E} = j^*E$ is also an electric field on $(S, \tilde{g})$ generated by some configuration of charges $\mathcal{C}' = \{(q_i', p_i)\}_{i=1}^{N'}$, where possibly $q_i' \neq q_i$. We do not necessarily assume either that $\tilde{g}$ is the inherited metric j^*g. Generally speaking, the answer is clearly negative, since actually $\tilde{E}$ need not be either divergence-free or a gradient field. The usual method for inducing a divergence-free vector field on S is due to Godbillon [15]. When there exists a submersive first integral $I : U \to \mathbb{R}$, U being some neighborhood of S, Godbillon's theorem ensures that $\tilde{E}$ is divergence-free in $S - \bigcup_{i=1}^{N'} p_i$ with respect to the volume form $\tilde{\Omega} = |\nabla I|^{-2} i_{\nabla I}\Omega$, where Ω stands for the volume element in (M, g). Note that $\tilde{\Omega}$ coincides with the volume element corresponding to $(S, \tilde{g})$ if one sets $\tilde{g} = |\nabla I|^{-2/(n-1)} j^*g$. However, since the charges are either attractors or repellers, there cannot exist any local first integrals differentiable at p_i, and thus the metric $\tilde{g}$ is not smooth.

The difficulties which arise can be easily understood with the following simple example. In dimension 2 the electric field is locally Hamiltonian, and a local first integral always exists. In the complex plane $(\mathbb{C}, \mathrm{d}z\mathrm{d}\bar{z})$, for example, the first integral of the electric field generated by a configuration $\mathcal{C} = \{q_i, z_i\}$ can be explicitly computed to yield

$$I(z, \bar{z}) = \frac{\operatorname{Re} \prod (z - z_i)^{q_i}}{\operatorname{Im} \prod (z - z_i)^{q_i}}.$$

Taking $\mathcal{C} = \{(-1, 1), (1, -1)\}$ and setting $z = x + \mathrm{i}y$, the first integral reads $I(x, y) = \frac{1}{2} y^{-1}(1 - x^2 - y^2)$, so the unit circle $S = I^{-1}(0)$ is invariant under the electric field. Nevertheless, I is not even continuous on the line $\{y = 0\}$, so the metric $|\nabla I|^{-2} j^*g = \csc^2\theta \mathrm{d}\theta^2$ is singular on S at the charges.

Proposition 7 provides an effective method for obtaining closed expressions for the potential function in certain spaces. Let $\tilde{V}$ be the potential created by a point charge situated at $\tilde{p}$ in a Riemannian manifold $(\tilde{M}, \tilde{g})$, and let G be a discrete group of isometries of $(\tilde{M}, \tilde{g})$ whose action on $\tilde{M}$ is free and properly discontinuous. Then the manifold $M = \tilde{M}/G$ inherits the complete analytic metric $g = \pi_* \tilde{g}$, where the analytic map $\pi : \tilde{M} \to M$ denotes the projection.

Let us suppose that there exists a sequence (c_a) such that the sum $\sum_{a \in G} [\tilde{V}(a\tilde{x}) - c_a]$ is finite for each $\tilde{x} \in \tilde{M}$. Then this sum takes the same value on each fiber $\pi^{-1}(x)$ and the analytic function $V : M \to \mathbb{R}$ defined by

$$V(x) = \sum_{a \in G} [\tilde{V}(a\tilde{x}) - c_a]$$

is independent of the choice of $\tilde{x} \in \pi^{-1}(x)$, and a Li–Tam potential created by a point charge situated at $p = \pi\tilde{p}$.

This approach is particularly convenient for studying the electric field on spaces of constant curvature, which can be obtained [20] by quotienting a space form $\mathbb{R}^n$, $\mathbb{H}^n$ or S^n by a discrete subgroup G of its isometry group, namely, $\mathrm{E}(n)$, $\mathrm{O}(n, 1)$ or $\mathrm{O}(n + 1)$ respectively. We shall illustrate this method with some examples.

Example 6. Let us consider the Euclidean plane $\mathbb{R}^2$, with Cartesian coordinates $-\infty < z, \theta < +\infty$, and the action of $\mathbb{Z}$ on $\mathbb{R}^2$ given by $\Theta_n(z, \theta) = (z, \theta + 2n\pi)$. In these coordinates, the metric of $\mathbb{R}^2$ takes the form $\mathrm{d}s^2 = \mathrm{d}z^2 + \mathrm{d}\theta^2$, and the potential created by a negative unit charge situated at $(0, 0)$ is given by $\tilde{V} = \frac{1}{4\pi} \log(z^2 + \theta^2)$.

Let us consider the flat cylinder $\mathbb{R} \times S^1 = \mathbb{R}^2/\mathbb{Z}$, with coordinates $-\infty < z < +\infty$, $-\pi < \theta < \pi$, and metric $\mathrm{d}s^2 = \mathrm{d}z^2 + \mathrm{d}\theta^2$. One can explicitly evaluate the potential of a negative unit charge situated at (0, 0) as

$$V = \frac{1}{4\pi} \sum_{n \in \mathbb{Z}} \log[z^2 + (\theta + 2\pi n)^2] - \log(1 + 4\pi^2 n^2)$$
$$= \frac{1}{4\pi} \log(\cosh z - \cos\theta) + \text{const.}$$

Obviously V tends to $+\infty$ at both ends since they are parabolic. The electric field is given by

$$E = -\frac{\sinh z \partial_z + \sin\theta \partial_\theta}{4\pi(\cosh z - \cos\theta)},$$

and tends to $\pm\frac{1}{4\pi}\partial_z$ as z tends to $\mp\infty$. The only critical point is $(0, \pi)$, and the circle $\{z = 0\}$ is invariant. The basin boundary is given by the invariant line $\{\theta = \pi\}$.

An analogous but more involved computation can be performed for other flat cylinders. For instance, the potential in $\mathbb{R}^3 \times S^1$, with coordinates (x, θ), is given by

$$V = -\frac{\sinh|x|}{32\pi^2|x|(\cosh|x| - \cos\theta)}.$$

Again the only critical point is $(0, \pi)$, and the boundary is the invariant plane $\{\theta = \pi\}$.

It is clear that this approach also works for a configuration of several charges on compact manifolds, provided that all the charges sum to zero as required by the argument in Proposition 4.

Example 7. Let us consider the Euclidean plane, with coordinates x, and the additive action of $\mathbb{Z}^2$ on $\mathbb{R}^2$. Let us define the symmetric sum

$$\sum_{n \in \mathbb{Z}^2} = \lim_{k \to \infty} \sum_{n_1 = -k}^{k} \sum_{n_2 = -k}^{k},$$

and consider an induced potential of the form

$$V = -\frac{1}{2\pi} \sum_{n \in \mathbb{Z}^2} \sum_{i=1}^{N} q_i \log|x - x_i + n| - c_n,$$

created by a configuration $\mathcal{C} = \{(q_i, x_i)\}$.

We consider first the case in which the charges in the configuration sum to zero, which without loss of generality can be reduced to $\mathcal{C} = \{(-1, 0), (1, -x_0)\}$. Since

$$V = -\frac{1}{2\pi} \sum_{n \in \mathbb{Z}^2} \log\frac{|x + x_0 + n|}{|x + n|} - c_n,$$

and

$$\log\frac{|x + x_0 + n|}{|x + n|} \sim \frac{x_0 \cdot (x + n)}{|x + n|^2} + \frac{\frac{1}{2}|x_0|^2|x + n|^2 - [x_0 \cdot (x + n)]^2}{|x + n|^4} + O(|n|^{-3}),$$

the choice

$$c_n = \frac{|x_0 \cdot n|^2}{1 + |n|^4} - \frac{|x_0|^2}{2 + 2|n|^2}$$

renders the sum for V uniformly convergent on compact sets not containing the charges, and thus leads to the potential created by two charges on the flat torus.

When the charges do not sum to zero, the potential cannot exist on any closed manifold by the elementary argument outlined in Proposition 4. This can be easily seen when considering potentials of the above form and the simplest configuration $\mathcal{C} = \{(-1, 0)\}$. As

$$\log |x + n| \sim \log |n| + \frac{x \cdot n}{|n|^2} + \frac{\frac{1}{2}|n|^2|x|^2 - (x \cdot n)^2}{|n|^4} + O(|n|^{-3}),$$

the best possible choice for c_n would be

$$c_n = \log |n| + \frac{(x_0 \cdot n)^2 - \frac{1}{2}|n|^2|x_0|^2}{|n|^4} + O(|n|^{-3})$$

for some fixed $x_0 \in \mathbb{R}^2$, which does not prevent the sum for V from diverging logarithmically but at $x = x_0$.

The same procedure can be applied to the space forms of positive or negative curvature. From the hyperbolic plane, whose fundamental solution was given in Example 4, one can obtain the electric field on the torus with g handles and negative curvature. From the round sphere, the potential created by two charges of magnitude ± 1 situated at antipodal points can be computed to yield

$$V = -c_n \int \csc^{n-1} r \, \mathrm{d}r,$$

where r denotes the geodesic distance to the positive charge, and can be used to study the electric field on spherical spaces [56].

7. Open problems

A major unanswered question in the study of electric fields on n-manifolds ($n \geq 3$) is that of proving or disproving that the basin boundary in $(\mathbb{R}^n, g)$ is always empty, which is equivalent to V having no critical points. This problem is physically relevant since $(\mathbb{R}^n, g)$ has a natural interpretation in Electrostatics as an anisotropic Euclidean space with dielectric tensor $\epsilon^{ij} = (\det g)^{1/2} g^{ij}$ [24].

In fact, we conjecture that the following stronger result also holds. Let $N_Z < \infty$ be the number of critical points of V in a space (M, g), V being the potential created by a point charge, and let $b_k(M)$ be the k-th Betti number of M. Then we conjecture that

$$N_Z \leq \sum_{k=1}^{n} b_k(M), \tag{10}$$

and the upper bound is saturated if and only if all the critical points are hyperbolic. In Section 4 it was proved that this conjecture holds when $n = 2$, but the proof relies on the classification of surfaces and on the particular properties of harmonic functions on 2-manifolds, and does not extend to higher dimensions.

Furthermore, we also believe that the electric field is generically Morse–Smale for an arbitrary number of charges, and hence structurally stable [35], so that the generic number of critical points of V is a topological invariant when only one charge is present.

A natural extension of this work is the study of the geometrical and topological properties of the dynamics of particles in a static electric field on Riemannian manifolds, and of coupled electric and magnetic fields which evolve in time according to the laws of special relativity.

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